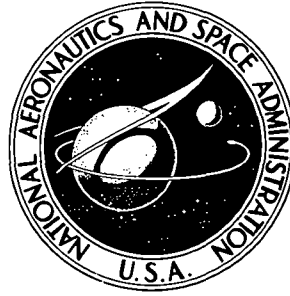


N74-10663

NASA TECHNICAL NOTE



NASA TN D-7313

NASA TN D-7313

# CASE FILE COPY

## INTERMEDIATE ENERGY PROTON-DEUTERON ELASTIC SCATTERING

*by John W. Wilson*

*Langley Research Center  
Hampton, Va. 23665*

1. Report No. NASA TN D-7313		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle  INTERMEDIATE ENERGY PROTON-DEUTERON ELASTIC SCATTERING				5. Report Date November 1973	
				6. Performing Organization Code	
7. Author(s) John W. Wilson				8. Performing Organization Report No. L-8915	
9. Performing Organization Name and Address  NASA Langley Research Center Hampton, Va. 23665				10. Work Unit No. 503-10-04-01	
				11. Contract or Grant No.	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546				13. Type of Report and Period Covered Technical Note	
				14. Sponsoring Agency Code	
15. Supplementary Notes					
16. Abstract  <p>A fully symmetrized multiple-scattering series is considered for the description of proton-deuteron elastic scattering. An off-shell continuation of the experimentally known two-body amplitudes that retains the exchange symmetries required for the calculation is presented. The one-boson exchange terms of the two-body amplitudes are evaluated exactly in this off-shell prescription. The first two terms of the multiple-scattering series are calculated explicitly whereas multiple-scattering effects are obtained as minimum variance estimates from the 146-MeV data of Postma and Wilson. The multiple-scattering corrections indeed consist of low-order partial waves as suggested by Sloan based on model studies with separable interactions. The Hamada-Johnston wave function is shown consistent with the data for internucleon distances greater than about 0.84 fm.</p>					
17. Key Words (Suggested by Author(s)) Proton Nucleon Deuteron Elastic scattering Multiple scattering				18. Distribution Statement Unclassified - Unlimited	
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 36	
				22. Price* Domestic, \$3.00 Foreign, \$5.50	

# INTERMEDIATE ENERGY PROTON-DEUTERON

## ELASTIC SCATTERING

By John W. Wilson  
Langley Research Center

### SUMMARY

A fully symmetrized multiple-scattering series is considered for the description of proton-deuteron elastic scattering. An off-shell continuation of the experimentally known two-body amplitudes that retains the exchange symmetries required for the calculation is presented. The one-boson exchange terms of the two-body amplitudes are evaluated exactly in this off-shell prescription. The first two terms of the multiple-scattering series are calculated explicitly whereas multiple-scattering effects are obtained as minimum variance estimates from the 146-MeV data of Postma and Wilson. The multiple-scattering corrections indeed consist of low-order partial waves as suggested by Sloan based on model studies with separable interactions. The Hamada-Johnston wave function is shown consistent with the data for internucleon distances greater than about 0.84 fm.

### INTRODUCTION

Although much attention has been given to the theoretical aspects of nucleon-deuteron elastic scattering, at intermediate energies, little progress has been made in understanding the cross section or polarization at backward angles (refs. 1 to 8). Multiple-scattering series have been applied with reasonable success where the first two terms (pickup and single-scattering) provide a good qualitative description of the data (refs. 1, 2, 7, 8, and 9). Quantitatively this success is largely limited to forward scattering which is reasonably determined by various approximations of the single-scattering integral (refs. 1 and 3 to 9). The usual pickup term is a complicated expression obtained by applying exchange symmetry to the direct multiple-scattering series (ref. 10). The "Born" approximation is used to simplify this term to a manageable form (ref. 7). It was believed for many years that the failure to obtain quantitative agreement near backward scattering

was due to the inadequacies of the Born approximation to the pickup term (refs. 1, 2, 6, 7, and 8). At intermediate angles others concluded (based on studies of double-scattering) that multiple-scattering corrections are important (refs. 9 and 11 to 13), but the question of multiple-scattering effects at backward angles was left unresolved since typical approximations prove inadequate at large momentum transfer. The importance of multiple scattering was further confused since successive terms of the multiple-scattering series are progressively more complicated so that explicit evaluation was made only for the simplest of models (for example, using separable interactions) or with many simplifying approximations in calculations with realistic models (refs. 3 and 11 to 15). Other corrections were suggested as of possible increasing importance toward intermediate and backward angles which include three-body effects (ref. 16), corrections to the impulse approximation (refs. 5, 14, and 17), off-shell effects (refs. 4, 5, 6, 9, and 17), and exchange symmetry effects other than pickup (refs. 7, 8, 15, and 18 to 21). At energies above a few hundred MeV, relativistic effects are expected to become important as well as resonance processes such as those exhibited in the Kerman and Kisslinger wave function (ref. 22) and the Craigie and Wilkin double-scattering mechanism (ref. 23).

A comprehensive treatment of nucleon-deuteron scattering must include an accurate description of the scattered nucleon polarization which is a sensitive function of the phase of the amplitude. At intermediate energy, the polarization is observed to have a maximum at about  $40^\circ$  followed by a minimum at  $100^\circ$  with a second relative maximum at about  $160^\circ$ , all in the center-of-mass system (refs. 18 and 24 to 28). The first maximum and the minimum in polarization as a function of scattering angle are reasonably accounted for by single scattering and are related to the phases of the nucleon-nucleon amplitudes in a sticking-factor approximation (refs. 5 to 8). A second maximum in polarization near backward angles appears as interference between the pickup and single-scattering amplitudes which is in qualitative agreement with experiment but is far too small in magnitude (refs. 7 and 8).

The backward scattering data are understood in terms of a simple model which is valid for intermediate energies. First, it is considered that nucleon exchange which arises from applying exchange symmetry to lowest order rearrangement collision (in distinction to pickup which is evaluated through Born approximation) is the principal contribution near backward scattering. Second, an accurate calculation of the single-scattering integral shows it to be almost two orders of magnitude below nucleon exchange (ref. 29) and therefore not an important contribution. Finally, an estimate of the magnitude of

double-scattering amplitude shows it to be nearly angle independent from  $130^\circ$  to  $180^\circ$  in the center of mass and a non-negligible contribution ( $\approx 15$  percent). Since the higher-order multiple scatterings are expected to be even more isotropic (ref. 14), the higher-order multiple scatterings contribute only to an S-wave. Since single scattering becomes increasingly small in the backward hemisphere, the model including only an S-wave plus nucleon exchange will describe the data. This model has polarization which necessarily vanishes at  $\cos \theta = -0.8$  and  $-1.0$  with a stationary value between. When the interference between the nucleon exchange and S-wave is destructive, a maximum in polarization is produced. This model (applicable only in the backward hemisphere) quantitatively explains the existing cross-section data and the second maximum in polarization from 77 MeV to 198 MeV. Although the zero crossing of polarization for the model ( $\theta \approx 142^\circ$ ) is somewhat different from the observed crossing ( $\theta \approx 135^\circ \pm 6^\circ$ ), the addition of a simple S-wave appears to be the missing ingredient needed to understand the data.

The essence of this S-wave is further clarified through the recent model considerations of Sloan (ref. 30). Sloan considers the three-particle scattering problem for particles which interact through a two-body isospin spin-dependent S-wave separable interaction. The parameters of the interaction are chosen to fit the low-energy nucleon-nucleon data. The interesting feature of the model is that it is exactly soluble. Sloan concludes on the basis of the model that the multiple-scattering series converges slowly at intermediate energies but that the slow convergence is restricted only to the S-wave amplitude. Furthermore, all partial waves with  $\ell > 2$  are defined by the nucleon exchange (not a Born approximation) plus single-scattering amplitudes with double scattering contributing only to the S-wave and P-wave amplitudes. That an S-wave amplitude should result from multiple scattering from spherically symmetric two-body interactions is perhaps not too surprising. The exact details of Sloan's calculations at intermediate energies are questionable since the two-body amplitudes in fact show a good deal of angular structure quite aside from questions about the correctness of a separable representation.

It is the purpose of the present paper to examine the adequacy of a multiple-scattering description of proton-deuteron scattering at intermediate energy. Although the multiple-scattering series is expected to converge slowly, model calculations show that the higher-order multiple-scattering terms contribute only to the low-order partial waves (ref. 30). The first two terms, neutron exchange and single scattering, are assumed to describe the high-order partial waves completely. Relativistic effects and resonance processes depend strongly on the depth at which the deuteron is probed and would appear

as modification in the higher-order partial wave amplitudes and therefore distinguishable from expected multiple-scattering effects. Three-body effects, which contribute to the scattering from only a small region of the deuteron interior (ref. 12), would contribute to S-wave scattering at intermediate energies, which makes them indistinguishable from expected multiple-scattering effects. Estimates of three-body effects show them to be small (refs. 12 and 16). In the present work, the deuteron is assumed to couple only to the nucleon channel and the internal structure is assumed adequately defined by a non-relativistic wave function. The proton-deuteron scattering is represented by a multiple-scattering series in which nucleon exchange and single scattering are calculated explicitly. The effects of various approximations of the single-scattering integral on the theory is considered. The totality of higher-order multiple scatterings is approximated by a phenomenological S-wave. The extent of the double-scattering contribution will be estimated by adding a P-wave to the model. The requirement of high-order phenomenological partial waves would be interpreted as evidence of possible resonance processes or relativistic effects not included in the present theory.

#### SYMBOLS

$a_{L_n}, b_{L_n}$	structure constants of deuteron wave function, $n = 1, 2, 3$
$D$	deuteron four momentum, amu
$D_2$	spin-doublet partial-wave amplitude, $\text{amu}^{-2}$
$E$	proton laboratory energy, MeV
$j_\beta$	nucleon current matrix element, $\beta = S, T, V, A, P$ , $\text{amu}^{-3}$
$M_D$	deuteron mass, amu
$m$	nucleon mass ( $\approx 1.0$ amu)
$P(\theta)$	nucleon polarization, dimensionless
$P_D$	deuteron D-state, percent

$p$	nucleon four momentum, amu
$p_1$	struck deuteron constituent-four momentum, amu
$p_2$	spectator deuteron constituent-four momentum, amu
$\vec{Q}$	momentum transfer in center of mass, amu
$Q_\ell$	spin-quartet partial-wave amplitude, amu <sup>-2</sup>
$s, t, u$	Mandelstam variables for two-nucleon system, amu <sup>2</sup>
$\hat{s}$	Lorentz invariant, amu <sup>2</sup>
$S, P$	S-wave and P-wave transition amplitudes, amu <sup>-2</sup>
$S_{L,L}(\vec{Q})$	deuteron sticking factor, dimensionless
$T$	transition matrix, amu <sup>-2</sup>
$T_s$	single-scattering transition matrix, amu <sup>-2</sup>
$T_x$	nucleon exchange transition matrix, amu <sup>-2</sup>
$t_\alpha$	two-body amplitudes in nucleon spin basis ( $\alpha = np, pp$ ), amu <sup>-2</sup>
$Y_L$	$2 \times 2$ representation of spin spherical harmonics with orbital angular momentum $L$ , dimensionless
$y_{JLS}^M$	usual spin spherical harmonic, dimensionless
$Z_2, Z_4$	uncoupled spin doublet and quartet amplitudes, dimensionless
$z$	off-shell two-body cosine of center-of-mass scattering angle, dimensionless

$\delta$	nucleon mass difference, off-shell parameter, $\text{amu}^2$
$\theta$	proton-deuteron center-of-mass scattering angle, degrees
$\vec{k}$	deuteron internal momentum, $\text{amu}$
$\mu$	projection of deuteron spin onto the Z-axis
$\vec{\pi}$	deuteron polarization vector, dimensionless
$\sigma(\theta)$	nucleon-deuteron elastic differential cross section, $\text{mb/sr}$
$\vec{\sigma}$	Pauli spin vector, dimensionless
$\varphi$	deuteron momentum space wave function, $\text{amu}^{-3/2}$
$\tau_{\beta}^{\alpha}$	Lorentz invariant two-body amplitudes for scattering from protons ( $\alpha = p$ ) or neutrons ( $\alpha = n$ ) in Current $\times$ Current basis, $\text{amu}^4$

**Subscripts:**

E	entering state
F	final state
L	deuteron orbital angular-momentum quantum numbers, dimensionless
$\ell$	proton-deuteron orbital angular-momentum quantum numbers, dimensionless
n	neutron
p	proton



Three vectors are denoted by  $\vec{\phantom{x}}$  above the quantity. Four vectors carry no special annotation. Dot products among four vectors  $b = (b^0, \vec{b})$  and  $c = (c^0, \vec{c})$  are as  $b \cdot c = b^0 c^0 - \vec{b} \cdot \vec{c}$ . Dagger ( $\dagger$ ) denotes Hermitian conjugation. Circumflex ( $\wedge$ ) denotes unit vector. Star ( $*$ ) denotes complex conjugation. The trace of the matrix A is denoted by  $\langle A \rangle$ .

## MULTIPLE SCATTERING AND IMPULSE APPROXIMATION

The form of the multiple-scattering series is shown in terms of Feynman graphs in figure 1 (refs. 12, 19, and 31). Higher-order multiple-scattering terms and three-body graphs (ref. 16) are not shown in the figure. The present form of the impulse approximation neglects the three-body graphs (ref. 32).

In the following p and D refer to the four momentum of the proton and deuteron, subscripts E and F refer to entering and final states, and  $\vec{\pi}(\mu)$  is the deuteron polarization for deuteron spin projection  $\mu$  along the z-axis in the deuteron rest frame. The natural units are chosen in which Planck's constant  $\hbar$  and the velocity of light c are unity. Energies are measured in atomic mass units, and the difference in proton and neutron mass is ignored ( $m_n = m_p \approx 1$  amu) except in calculating the deuteron binding energy.

The contributions of the graphs of figure 1 will be used to calculate the transition amplitude, and they are obtained in the nonrelativistic limit in a manner outlined by Gross (ref. 32). The result is

$$T = T_X + T_S + S + P \quad (1)$$

where the four contributions are

Nucleon exchange:

$$T_X = \left( |B| + \vec{\kappa}_E^2 \right) \varphi \left( \vec{\pi}_E, \vec{\kappa}_E \right) \varphi \left( \vec{\pi}_F^*, \vec{\kappa}_F \right) \quad (2)$$

where

$$\vec{\kappa}_{E,F} = \vec{p}_{E,F} - \frac{1}{2} D_{E,F}$$

$$B = M_D - m_p - m_n$$

Single scattering:

$$T_S = \int \sum_{\alpha=p,n} \sum_{\beta=S,T,V,A,P} \tau_{\beta}^{\alpha}(s, t, u, \delta) j_{\beta}(p_F, p_E) \left\langle \varphi(\vec{\pi}_E, \vec{\kappa}_E) \varphi(\vec{\pi}_F, \vec{\kappa}_F) j^{\beta}(p_{1F}, p_{1E}) \right\rangle d^3\kappa \quad (3)$$

where

$$\vec{\kappa}_E = \vec{\kappa} + \vec{Q}/4$$

$$\vec{\kappa}_F = \vec{\kappa} - \vec{Q}/4$$

$$p_{1E,F} = \frac{1}{2} D_{E,F} + \vec{\kappa}_{E,F}$$

Higher order:

$$S = D_0 Z_2(\vec{\pi}_F^*, \vec{\pi}_E) + Q_0 Z_4(\vec{\pi}_F^*, \vec{\pi}_E) \quad (4a)$$

$$P = D_1 \cos \theta Z_2(\vec{\pi}_F^*, \vec{\pi}_E) + Q_1 \cos \theta Z_4(\vec{\pi}_F^*, \vec{\pi}_E) \quad (4b)$$

with the following explanation of the functions and variables.

The first graph on the left in figure 1 is the well-known nucleon exchange term which results as a direct application of exchange symmetry in the lowest-order rearrangement graph (order refers to the number of proper vertices). This term contributes to the scattering in the elastic channel only because the deuterons arising from direct and rearrangement collisions are experimentally indistinguishable. This amplitude has been previously derived through Feynman analysis (ref. 33), and is usually expressed in terms of the Blankenbecker-Cook amplitude (refs. 32 to 35), which has in turn been related to the nonrelativistic wave function by Gross (ref. 32). Thus, the result is equation (2) in the nonrelativistic limit with  $\varphi(\vec{\pi}, \vec{\kappa})$  representing the wave function (to be discussed more fully) of a deuteron with polarization  $\vec{\pi}$  and relative internal momentum  $\vec{\kappa}$ . The pickup term of potential theory arises from applying exchange symmetry to the direct multiple-scattering series terms (ref. 10). A term equal to the nucleon exchange term is obtained as a Born approximation to the pickup amplitude (refs. 7, 8, and 14).

The next two graphs in figure 1 represent the scattering of the incident projectile by a single constituent of deuterium. All vertices are properly symmetrized under exchange of identical nucleons. The single-scattering amplitude has been previously derived

through Feynman analysis (refs. 11, 12, and 19). The nonrelativistic reduction follows closely the work of Gross (ref. 32). The nucleon-nucleon scattering amplitude is retained in fully relativistic form. The two-body amplitudes are represented as (refs. 36 and 37):

$$t_{\alpha}(p_{1F}, p_{2F}; p_{1E}, p_{2E}) = \sum_{\beta=S,T,V,A,P} \tau_{\beta}^{\alpha}(s,t,u,\delta) j_{\beta}(p_{1F}, p_{1E}) \times j^{\beta}(p_{2F}, p_{2E})$$

where  $\beta$  is summed over the five  $\beta$ -decay type currents; Scalar, Tensor, Vector, Axial-vector, and Pseudo-scalar. A similar Current  $\times$  Current representation appears in reference 19 by Chahoud and others. The  $s$ ,  $t$ , and  $u$  variables are the usual Mandelstam variables and  $\delta$  is the difference of mass squared of the struck constituent in the entering and final state. The two-body amplitudes are the symmetrized nucleon-nucleon amplitudes. This symmetry is expressed through

$$\tau_{\beta}^I(s,t,u,\delta) = (-1)^I \sum_{\alpha} \Gamma_{\beta\alpha} \tau_{\alpha}^I(s,u,t,\delta)$$

where  $I = 0, 1$  is the isospin state and  $\Gamma_{\beta\alpha}$  is the Fierz transformation. Clearly any off-shell approximation should preserve this symmetry (that is, the Pauli principle). The continuation of experimental data to off-shell values is given in a later section (see also ref. 29). A term similar to single scattering results from applying exchange symmetry to rearrangement processes in potential theory (refs. 30 and 38).

The totality of higher-order terms is represented by the S-wave of equation (4a). This representation is in accord with the work by Sloan (ref. 30) who showed through model calculations that triple and higher-order terms in the multiple-scattering series converge slowly and contribute to an S-wave. The double scattering is suggested by Sloan to contribute most to the S-wave but with a few-percent contribution to the P-wave of equation (4b). The P-wave contributions coupling the doublet and quartet spin states are neglected. The two uncoupled spin transition amplitudes are given by

Doublet

$$Z_2(\vec{\pi}_F^*, \vec{\pi}_E) = \frac{1}{3} \left[ \vec{\pi}_F^* \cdot \vec{\pi}_E + i\vec{\sigma} \cdot (\vec{\pi}_F^* \times \vec{\pi}_E) \right] \quad (5)$$

Quartet

$$Z_4(\vec{\pi}_F^*, \vec{\pi}_E) = \frac{1}{3} \left[ 2\vec{\pi}_F^* \cdot \vec{\pi}_E - i\vec{\sigma} \cdot (\vec{\pi}_F^* \times \vec{\pi}_E) \right] \quad (6)$$

where  $\vec{\sigma}$  is the ordinary Pauli spin operator. The  $D_\ell$  and  $Q_\ell$  of equations (4a) and (4b) are complex functions of center-of-mass energy which are presently uncalculated. They will be determined so as to give best fits to the experimental data.

Calculation of proton-deuteron elastic scattering in an impulse approximation requires, as inputs, information concerning the two-nucleon system, that is, the deuteron wave function and two-body scattering amplitudes (compare eqs. (2) and (3)). For the energy range considered here, these inputs are relatively well known from more than three decades of experimental work. Much of this experimental information is summarized in recent phenomenological potentials (refs. 39 to 42) and phase-shift analysis (refs. 43 to 45).

### DEUTERON WAVE FUNCTION

The most general form of the deuteron wave function is implied from the following invariant operations: translation, rotation, space inversion, time inversion, and nucleon exchange (Pauli). From translational invariance, it is concluded that the wave function of a deuteron of momentum  $\vec{D}$ , polarization  $\vec{\pi}$ , projected onto a two-nucleon state has the representation

$$\Psi_{\nu_1\nu_2}(\vec{x}_1, \vec{x}_2, \vec{D}, \vec{\pi}) = (2\pi)^{-3/2} e^{-i\vec{x} \cdot \vec{D}} \chi_{\nu_1\nu_2}(\vec{\pi}, \vec{x}_1, \vec{x}_2) \quad (7)$$

with

$$\chi_{\nu_1\nu_2}(\vec{\pi}, \vec{x}_1, \vec{x}_2) = (2\pi)^{-3/2} \int d^3k e^{-i\vec{r} \cdot \vec{k}} \Psi_{\nu_1\nu_2}(\vec{\pi}, \vec{p}_1, \vec{p}_2) \quad (8)$$

and

$$\vec{x} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$$

$$\vec{r} = \vec{x}_1 - \vec{x}_2$$

$$\vec{k} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$$

where  $\vec{x}_1$  and  $\vec{x}_2$  are the position vectors of the two nucleons and  $\nu_1$  and  $\nu_2$  are their spin projections.

Considered as a  $2 \times 2$  matrix,

$$\begin{bmatrix} \psi_{\nu_1 \nu_2}(\vec{\pi}, \vec{p}_1, \vec{p}_2) \end{bmatrix} \equiv \varphi(\vec{\pi}, \vec{p}_1, \vec{p}_2) c^{-1} \quad (9)$$

with

$$c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

It is found that  $\varphi(\vec{\pi}, \vec{p}_1, \vec{p}_2)$  has the following form in the deuteron rest frame ( $\vec{p}_1 = -\vec{p}_2 = \vec{k}$ ) due to the remaining invariances:

$$\varphi(\vec{\pi}, \vec{k}) = \varphi_0(\vec{k}) Y_0(\vec{\pi}) - \varphi_2(\vec{k}) Y_2(\vec{\pi}, \vec{k}) \quad (10)$$

with

$$\sqrt{4\pi} Y_0(\vec{\pi}) = \vec{\pi} \cdot \vec{\sigma} / \sqrt{2}$$

$$\sqrt{4\pi} Y_2(\vec{\pi}, \vec{k}) = \frac{3}{2} \vec{\pi} \cdot \hat{k} \hat{k} \cdot \vec{\sigma} - \frac{1}{2} \vec{\pi} \cdot \vec{\sigma}$$

The matrices  $Y_0$  and  $Y_2$  are related to the ordinary spin spherical harmonics by

$$\sum_{\nu_1 \nu_2} \left\{ Y_0[\vec{\pi}(\mu)] c^{-1} \right\}_{\nu_1 \nu_2} \chi_{\nu_1} \chi_{\nu_2} = \mathcal{Y}_{101}^{\mu}(\hat{k}) \quad (11)$$

$$\sum_{\nu_1 \nu_2} \left\{ Y_2[\vec{\pi}(\mu), \vec{k}] c^{-1} \right\}_{\nu_1 \nu_2} \chi_{\nu_1} \chi_{\nu_2} = -\mathcal{Y}_{121}^{\mu}(\hat{k}) \quad (12)$$

where  $\chi_{1/2} = \alpha$ ,  $\chi_{-1/2} = \beta$  is in agreement with the representations of Gross (ref. 32) and Gourdin and others (ref. 34). It is concluded from equations (10), (11), and (12) that

$$\varphi_0(\kappa) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} j_0(\kappa r) u(r) r dr$$

$$\varphi_2(\kappa) = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} j_2(\kappa r) w(r) r dr$$

where  $u(r)$  and  $w(r)$  are the ordinary S and D state radial wave functions.

Various solutions for  $u(r)$  and  $w(r)$  are offered for different phenomenological potentials (refs. 40, 41, 42, and 46). The solutions to the Hamada-Johnston potential found with variational methods by Humberston and Wallace (ref. 46), which are presented in a convenient analytic form, will be used here. The Fourier transform of their analytic wave functions (ref. 46) can be written as sums of exponential integrals of complex argument as used in the present calculations.

## TWO-BODY TRANSITION AMPLITUDES

The second input to the impulse approximation is the nucleon-nucleon scattering amplitude. Although this amplitude is not known from fundamentals, it has been the object of much experimental work and is reasonably well known to a few hundred MeV with uncertainties entering at higher energies due to lack of experimental data and knowledge of inelastic processes (ref. 44).

The on-shell two-body amplitudes  $\tau_{\beta}^{\alpha}(s,t)$  are calculated from the Livermore phase shifts (ref. 45) but must be continued off-shell before the integral, equation (3), can be evaluated.

The invariant off-shell amplitudes are taken as functions of the three Mandelstam variables  $s$ ,  $t$ , and  $u$  with a fourth variable  $\delta$  as the mass difference of the struck nucleon in the entering and final state

$$s = (p_E + p_{1E})^2$$

$$t = (p_F - p_E)^2$$

$$u = (p_{1F} - p_E)^2$$

$$\delta = p_{1F}^2 - p_{1E}^2$$

In general, the values of the four invariants are independent and

$$s + t + u \neq 4m^2$$

The on-shell-like variables  $\hat{s}$  and  $\delta_0$  defined by

$$\hat{s} + t + u \equiv 4m^2$$

$$\delta_0 = 0$$

are introduced, and the invariant amplitude is expanded in a Taylor series in the neighborhood of the on-shell region

$$\tau_{\beta}^{\alpha}(s, t, u, \delta) = \tau_{\beta}^{\alpha}(\hat{s}, t, u, \delta_0) + (s - \hat{s}) \frac{\partial}{\partial s} \tau_{\beta}^{\alpha}(\hat{s}, t, u, \delta_0) + (\delta - \delta_0) \frac{\partial}{\partial \delta} \tau_{\beta}^{\alpha}(\hat{s}, t, u, \delta_0) + \dots$$

For the present, only the first term of this series is retained which is the experimentally known amplitude. This off-shell continuation evaluates exactly the one-boson exchange parts of the amplitudes and preserves the Pauli principle in the continuation (for more detail see ref. 29). The errors in neglecting the second term were shown to be small (at least for quasi-elastic scattering) at intermediate energy through explicit evaluation (ref. 17) while the third term must vanish if time reversal invariance is to be preserved.

## EVALUATION OF THE SINGLE-SCATTERING INTEGRAL

The principal assumption normally used in developing approximations to single scattering, equation (3), is that the product of two-body amplitudes and spin spherical harmonics is a slowly varying function of internal momentum compared to the radial-wave functions. Taylor series expansions of two-body amplitudes and spin spherical harmonics are then made with only the first term retained, leaving an integral over radial-wave functions (the sticking factors). Various approximations are obtained by the choice of expansion point. The only approximation of this type considered here is the Kottler-Kowalski prescription (ref. 7). An alternative is to examine the radial-wave function products which have many relative extrema. The two-body amplitudes and spin spherical harmonics are then expanded about each extremum and multiplied by appropriate weight determined from the deuteron wave functions (refs. 29 and 47). This procedure results in an approximation which combines sums of two-body amplitudes and spin spherical harmonics evaluated at points of principle contribution to the integral, equation (3). A third method is to perform a numerical integration in which a convergence test is used as a standard for adequacy of approximation (refs. 29 and 47).

The implementation of the above approximations of the integral in equation (3) is aided by taking the deuteron wave function as a sum of Gaussians

$$\varphi_L(\kappa) = \sum_n a_{L_n} \exp(-b_{L_n} \kappa^2) \quad (13)$$

The coefficients given in table I are fits to the Hamada-Johnston wave functions (ref. 46) and are reasonable fits to 0.28 fm. It is seen from figure 2 that the S-state is least accurately approximated. As a check on the effect of these errors on the integral in equation (3), the sticking factors for the S-state are compared in table II to the analytic sticking factors derived from the Humberston-Wallace  $L = 4$ ,  $N = 2$  analytic wave functions (ref. 46) and the values computed by Glendenning and Kramer for their potential number 9 (ref. 48).

The three approximations of the integral in equation (3) are represented as follows: Let the product of two-body amplitudes and spin spherical harmonics be combined into the symbol

$$\begin{aligned} \tau(s, t, u, \delta)_{L_E L_F} &= \sum_{\alpha=p,n} \sum_{\beta=S,T,V,A,P} \tau_{\beta}^{\alpha}(s, t, u, \delta) j_{\beta}(p_F, p_E) \\ &\times \left\langle Y_{L_E}(\vec{\pi}_E, \vec{\kappa}_E) Y_{L_F}(\vec{\pi}_F^*, \vec{\kappa}_F) j^{\beta}(p_{1_F}, p_{1_E}) \right\rangle \end{aligned} \quad (14)$$

so that the single-scattering integral in equation (3) becomes

$$\begin{aligned} T_S &= \sum_{L_E L_F} \int \tau(s, t, u, \delta)_{L_E L_F} \varphi_{L_E}(\kappa_E) \varphi_{L_F}(\kappa_F) d^3 \kappa \\ &= \sum_{\substack{L_E L_F \\ n \quad m}} a_{L_E n} a_{L_F m} \exp\left[-\alpha(1 - \beta^2) \vec{Q}^2/16\right] \int \tau(s, t, u, \delta)_{L_E L_F} \exp\left[-\alpha\left(\vec{\kappa} - \frac{1}{4}\beta\vec{Q}\right)^2\right] d^3 \kappa \end{aligned} \quad (15)$$

where the last equality follows from use of the Gaussian wave functions and

$$\alpha = b_{L_F m} + b_{L_E n} \quad (16)$$

$$\beta = (b_{L_F m} - b_{L_E n})/\alpha \quad (17)$$



The following approximations may now be written:

Kottler-Kowalski (ref. 7)

$$T_S \approx \sum_{L_E L_F} \tau(s, t, u, \delta)_{L_E L_F} S(\vec{Q})_{L_E L_F} \quad (18)$$

where the sticking factor is given by

$$\begin{aligned} S(\vec{Q})_{L_E L_F} &= \int \varphi_{L_E} \left( \vec{k} - \frac{1}{4} \vec{Q} \right) \varphi_{L_F} \left( \vec{k} + \frac{1}{4} \vec{Q} \right) d^3 k \\ &= \sum_{n, m} a_{L_E n} a_{L_F m} \left( \frac{\pi}{\alpha} \right) \exp \left[ -\alpha (1 - \beta^2) \vec{Q}^2 / 16 \right] \end{aligned} \quad (19)$$

Asymptotic method

$$T_S \approx \sum_{\substack{L_E L_F \\ n \ m}} a_{L_E n} a_{L_F m} \left( \frac{\pi}{\alpha} \right) \exp \left[ -\alpha (1 - \beta^2) \vec{Q}^2 / 16 \right] \tau(s, t, u, \delta)_{L_E L_F} \Big|_{\vec{k} = \frac{1}{4} \beta \vec{Q}} \quad (20)$$

Numerical integration

The variables chosen for the numerical integration are for a cylindrical system with  $\hat{z}$  along  $\vec{Q}$ . This choice reduces equation (3) to a form for direct application of Gauss integration by using Hermite, Laguerre, and Legendre polynomials for components of  $\vec{k}$  along  $\vec{Q}$ , for components perpendicular to  $\vec{Q}$ , and for the angular variable, respectively.

The magnitude squared of the single-scattering integral for the three approximations is shown in figure 3. There is a small error for the Kottler-Kowalski and Asymptotic-method approximations in the forward hemisphere due to inadequate approximation of the integral over the D-state spin spherical harmonic. The principal errors in the backward hemisphere are due to the rapid variation of the two-body amplitudes which arise from exchange symmetry terms in the two-body scattering (refs. 29 and 47). The main result from accurate evaluation of single scattering is the continued decrease of the amplitude in the backward hemisphere. The errors of low-order approximations have been incorrectly interpreted by others as the mechanism producing the backward scattering peak at higher energies (refs. 19 and 20).

## ANALYSIS OF EXPERIMENTAL DATA

The proton-deuteron elastic scattering amplitude (eq. (1)) is related to the differential cross section  $\sigma(\theta)$  and the proton polarization produced by the scattering  $P(\theta)$  by

$$\sigma(\theta) = (2\pi)^4 C \left( \frac{m_p M_D}{m_p + M_D} \right)^2 \frac{1}{6} \sum_{\mu_E \mu_F} \langle T^\dagger T \rangle \quad (21)$$

$$P(\theta) = \frac{\sum_{\mu_E \mu_F} \langle T^\dagger \sigma_z T \rangle}{\sum_{\mu_E \mu_F} \langle T^\dagger T \rangle} \quad (22)$$

where  $\langle \rangle$  denotes trace over nucleon spin indices and  $C = 10(0.2101)^2$  is a conversion factor from  $\text{amu}^{-2}$  to mb. The theory includes parameters (principally the S and P wave contributions of higher-order multiple scattering) which will be adjusted to obtain best agreement with the data. These parameters are estimated by minimizing

$$\begin{aligned} F(\hat{P}) = & \sum_{i=1}^{\frac{1}{2}N_x} \left\{ \left[ \sigma(\theta_i) - N_\sigma \sigma_{\text{exp}}(\theta_i) \right] / \Delta \sigma_{\text{exp}}(\theta_i) \right\}^2 \\ & + \sum_{i=1}^{\frac{1}{2}N_x} \left\{ \left[ P(\theta_i) - N_p P_{\text{exp}}(\theta_i) \right] / \Delta P_{\text{exp}}(\theta_i) \right\}^2 \end{aligned} \quad (23)$$

where  $\hat{P}$  is the parameter estimate, the quoted experimental uncertainties ( $\Delta \sigma_{\text{exp}}$  and  $\Delta P_{\text{exp}}$ ) are chosen as weights, and the uncertainty due to finite detector size is ignored. The factors  $N_\sigma$  and  $N_p$  allow for renormalization of experimental data and relate to the uncertainty of initial beam intensities (unpolarized and polarized) used in the experiment which are often known to no better than 10 percent.

Assuming the experimental errors (for fixed normalizations) to be normally distributed with standard deviation given by the quoted experimental uncertainty makes the minimum  $F(\hat{P})$  to be bounded by a chi-square variable (actually,  $F(\hat{P}) \approx \chi^2$ ) with the number of degrees of freedom equal to the number of values of cross section and polarization combined. For  $N_x$  (number of data points) large, the test for goodness of fit requires

$$N_x - 2\sqrt{2N_x} \leq F(\hat{P}) \leq N_x + 2\sqrt{2N_x}$$

for the fit to be acceptable. Standard minimization procedures were used to find the minimum.

## RESULTS AND DISCUSSION

The principal parameters in the present model are the S-wave and P-wave amplitudes of equation (1). In addition, the D-state percentage is taken as a parameter in the model in an attempt to extract it from the experimental data. The shape of the radial wave function is taken to be that of Hamada and Johnston (ref. 46). The parameters are estimated from the 146-MeV cross-section and polarization data of Postma and Wilson (ref. 18). The polarization renormalization is taken as  $N_p = 0.93$  in accord with references 25 and 49. The cross-section renormalization  $N_\sigma$  was taken as a parameter and is to be compared to the latest quoted value  $N_\sigma = 0.92$  (ref. 26).

The data analysis was first guided by the suggestion of Sloan (ref. 30) that higher-order multiple scattering is predominantly S-wave with double scattering contributing a small P-wave (few percent) amplitude. The initial fits ignored the P-wave completely. The data were fit using only the S-wave of equations (1) and (21) to (23) for the three approximations of the single-scattering integral. The resulting fitted parameters are given in table III with the resulting curves in figures 4 through 6. The experimental values (rectangles in figures) plotted in the figures are not those of the originators (ref. 18) but have been renormalized by  $N_\sigma = 0.92$  (ref. 26) and  $N_p = 0.93$  (refs. 25 and 49). It is seen from the values of  $F(\hat{P})$  obtained that the more accurate approximations of single scattering provide progressively worse agreement with the data. This fact can also be observed in the visual comparisons, especially at backward angles, in figures 4 through 6. The deterioration of the fits for the more accurate approximations occurs since backward single scattering interferes destructively with the nucleon exchange amplitude. A more accurate approximation of single scattering diminishes its magnitude (see fig. 3), hence reducing the interference term and thus providing worse fits to the data. The values of  $F(\hat{P})$  for each of these three fits are outside the acceptable range given by  $60 \pm 22$ . The assumption that higher-order multiple scattering is mostly S-wave must be rejected.

The fact that higher-order multiple scatterings are not predominantly S-wave as predicted by Sloan (ref. 30) is not too surprising since his predictions are based on the

assumption that the two-body amplitudes are isotropic and separable. Such a representation of the two-body amplitudes is not realized in nature. It is believed that the P-wave contribution from double scattering predicted by Sloan is underestimated and reasonable fits should be obtained by adding a non-negligible P-wave to the model. This possibility is contingent on relativistic effects and resonance processes being negligible at this energy since their first-order effects are expected to modify high-order partial-wave amplitudes.

The curves in figure 7 show the effect of adding the uncoupled spin P-wave to the formalism. It is seen in table III that the minimum of the fitting function drops to a value which indicates that the average data point is about  $1\sigma$  distance from the theoretical curve. The parameters were fit again using the numerical integration for the single-scattering integral with P-wave included but with the D-state percentage restricted to  $6.93 \pm 1.0$  and  $6.93 \pm 1.5$ , and the respective results are shown in the last two lines of table III. The extreme ( $P_D = 5.93$ ) of these last two results is compared in figure 8 with the experimental data. Comparison of the results in figures 7(a) and 8(a) shows the major adjustment to be in the cross sections between  $140^\circ$  and  $170^\circ$  scattering angles. A modest improvement in cross section between  $40^\circ$  and  $80^\circ$  for the higher  $P_D$  is also noted. It is impossible to distinguish on the basis of a chi-square test between these last three fits.

Examination of table III and the visual comparison of figures 6 through 8 show that the addition of a large P-wave contribution to the model is unavoidable. A rather large range of D-state percentage ( $3.93 \leq P_D \leq 5.93$ ) is shown consistent with the data. That the D-state percentage is poorly determined is due to correlations among the fitted parameters and the fact that the D-state contribution is a second-order effect at this energy. More accurate cross sections in the range of  $40^\circ$  to  $80^\circ$  and  $140^\circ$  to  $170^\circ$  could possibly help to reduce the range of  $P_D$  over which good fits can be obtained (compare figs. 7(a) and 8(a)). An alternative may be to calculate explicitly the main contribution of the S- and especially the P-wave amplitudes (that is, double scattering) and thereby eliminate the principle part of the correlations in the fits. However, such a calculation may not be possible with full spin dependencies on present-day computers unless some innovative approximations are found.

Finally, the extent of the multiple-scattering effects are summarized in figure 9 by comparing the partial cross sections of nucleon exchange plus single scattering with the multiple-scattering contribution estimated from the data (that is, S + P). It is seen from figure 9 that the multiple scattering has two broad maxima at forward and backward angles. The cross section is dominated by multiple scattering in the region  $110^\circ$  to  $140^\circ$ .

Presumably, the angular structure of the multiple-scattering amplitude would be given by double scattering. It would be interesting to see whether a meaningful calculation of double scattering would give such results.

### CONCLUDING REMARKS

The present belief in the correctness of a multiple-scattering series formulation of proton-deuteron scattering lies in the fact that the first few terms plus low-order partial waves provide a model in good agreement with the data. The fact that high-order phenomenological partial waves are not required indicates that relativistic corrections and resonance process are not important at the present energy. The low-order partial waves are understood to be higher-order multiple-scattering terms on the basis of model calculations of Queen and especially Sloan. The large phenomenological P-wave contribution to the amplitude in comparison to the few percent predicted by Sloan is believed due to the much stronger angular variation of the real two-body amplitudes as opposed to the S-wave two-body interaction in Sloan's model. The clarity of current understanding lies in the present structure of the multiple-scattering formalism in which nucleon exchange, fully symmetrized single scattering, etc., arise in a natural way from Feynman analysis of the three-body amplitude. In distinction, the pickup term is usually obtained as a Born approximation to exchange symmetry terms of the usual direct multiple-scattering amplitude. This procedure led to the widely accepted conclusion that inadequacies of the theory were indeed a problem of the Born approximation. The present result eliminates this possibility and clearly indicates higher-order multiple scattering as the necessary ingredient to complete the theory provided, of course, that terms are properly symmetrized.

It has been shown that the D-state is poorly determined from the presently examined experimental data. This poor determination occurs because the range of momentum over which the deuteron is presently probed is mostly the region where the S-state contributions are most important (Internal momentum  $\leq 0.25$  amu). Those regions where the D-state is most effective in nucleon exchange and single scattering are also the regions where multiple-scattering effects are very important. Although there is modest improvement in cross section in the angular range  $40^\circ$  to  $80^\circ$  in going to higher values of D-state percentage, both values of D-state percentage are seen to be within the experimental-error limits. A positive conclusion from the fits is that the Hamada-Johnston S-state wave function is shown consistent with the data to distances beyond about 0.84 fm.

Although it is gratifying to see that a multiple-scattering theory of proton-deuteron scattering is nearly at hand, at least at intermediate energy, it still remains to be demonstrated that inclusion of explicitly calculated higher-order terms will bring about improvement of the theory. Even if improved calculations are made, it currently appears that little additional information about the deuteron wave function will be learned at the presently considered energy. Perhaps the most important result from the present study is an estimate based on the Postma and Wilson experiments of the multiple-scattering effects to which future calculations will hopefully converge.

Langley Research Center,  
National Aeronautics and Space Administration,  
Hampton, Va., July 24, 1973.

## REFERENCES

1. Chew, Geoffrey F.: High Energy Elastic Proton-Deuteron Scattering. *Phys. Rev.*, second ser., vol. 84, no. 5, Dec. 1, 1951, pp. 1057-1058.
2. Chew, G. F.; and Goldberger, M. L.: The Production of Fast Deuterons in High Energy Nuclear Reactions. *Phys. Rev.*, second ser., vol. 77, no. 4, Feb. 15, 1950, pp. 470-475.
3. Glauber, R. J.: Cross Sections in Deuterium at High Energies. *Phys. Rev.*, second ser., vol. 100, no. 1, Oct. 1, 1955, pp. 242-248.
4. Kerman, A. K.; McManus, H.; and Thaler, R. M.: The Scattering of Fast Nucleons From Nuclei. *Ann. Phys. (N.Y.)*, vol. 8, no. 4, Dec. 1959, pp. 551-635.
5. Kowalski, K. L.; and Feldman, D.: Elastic Nucleon-Deuteron Scattering. *Phys. Rev.*, second ser., vol. 130, no. 1, Apr. 1, 1963, pp. 276-291.
6. Benoist-Gueutal, P.; and Gomez-Gimeno, F.: Diffusion Élastique Nucléon-Deuton a 155 MeV Interprétation Théorique. *Phys. Lett.*, vol. 13, no. 1, Nov. 1, 1964, pp. 68-69.
7. Kottler, H.; and Kowalski, K. L.: D-State and Pickup Effects in Nucleon-Deuteron Scattering in the Impulse Approximation. *Phys. Rev.*, second ser., vol. 138, no. 3B, May 10, 1965, pp. B619-B626.
8. Benoist-Gueutal, P.; and Gomez-Gimeno, F.: Calcul de la Polarisation des Protons dans la Diffusion Élastique p-d à 150 MeV. *J. Phys.*, vol. 26, July 1965, pp. 403-404.
9. L'Huillier, M.: Approximations dans le Calcul de la Diffusion Élastique Proton-Deutéron à Moyenne Énergie. *Nucl. Phys.*, vol. A122, no. 3, Dec. 30, 1968, pp. 667-674.
10. Kowalski, K. L.: Identity Effects in Multiple-Scattering Theories. *Nuovo Cimento*, vol. XXX, no. 1, Oct. 1, 1963, pp. 266-277.
11. Harrington, David R.: Double Scattering Corrections to High-Energy Diffraction Scattering From Deuterons. *Phys. Rev.*, second ser., vol. 135, no. 2B, July 27, 1964, pp. B358-B364.

12. Pumplin, Jon: Elastic Scattering From Deuterium. *Phys. Rev.*, second ser., vol. 173, no. 5, Sept. 25, 1968, pp. 1651-1659.
13. Remler, Edward A.: Corrections to the Glauber Approximation. *Bull. Amer. Phys. Soc.*, ser. II, vol. 13, no. 4, Apr. 22, 1968, p. 676.
14. Queen, N. M.: Corrections to the Impulse Approximation for Elastic Nucleon-Deuteron Scattering. *Nucl. Phys.*, vol. 55, June 1964, pp. 177-196.
15. Remler, Edward A.: High-Energy Scattering by Nuclei. *Phys. Rev.*, second ser., vol. 176, no. 5, Dec. 25, 1968, pp. 2108-2112.
16. Harrington, David R.: Three-Body Interaction Effects in High-Energy Scattering From Deuterons. *Phys. Rev.*, second ser., vol. 176, no. 5, Dec. 25, 1968, pp. 1982-1985.
17. Everett, Allen: Corrections to the Impulse Approximation in Scattering From Deuterons. *Phys. Rev.*, second ser., vol. 126, no. 2, Apr. 15, 1962, pp. 831-845.
18. Postma, Herman; and Wilson, Richard: Elastic Scattering of 146 Mev Polarized Protons by Deuterons. *Phys. Rev.*, second ser., vol. 121, no. 4, Feb. 15, 1961, pp. 1229-1244.
19. Chahoud, J. N.; Russo, G.; Mazzanti, P.; and Salvini, M.: Dispersive Impulse Approximation and Nucleon-Deuteron Elastic Scattering. *Nuovo Cimento*, ser. 10, vol. LVIA, no. 4, Aug. 21, 1968, pp. 838-866.
20. Kanada, Hinoyuki; Sakai, Kenji; and Yasuno, Masaru: Backward Peak in Proton-Deuteron Elastic Scattering at High Energy. *Progr. Theor. Phys.*, vol. 41, no. 6, June 1969, pp. 1605-1606.
21. Bertocchi, L.; and Capella, A.: Double-Scattering Model for High-Energy Backward Proton-Deuteron Scattering. *Nuovo Cimento*, vol. LI A, no. 2, Sept. 21, 1967, pp. 369-400.
22. Kerman, A. K.; and Kisslinger, L. S.: High-Energy Backward Elastic Proton-Deuteron Scattering and Baryon Resonances. *Phys. Rev.*, second ser., vol. 180, no. 5, April 25, 1969, pp. 1483-1489.
23. Craigie, N. S.; and Wilkin, C.: Elastic Proton-Deuteron Scattering at Large Angles. *Nucl. Phys. B*, vol. B14, no. 3, Dec. 15, 1969, pp. 477-488.



24. Poulet, M.; Michalowicz, A.; Kuroda, K.; and Cronenberger, D.: Mesure des Paramètres P, D, R et R' dans la Diffusion Élastique Proton-Deuton à 135 MeV. Nucl. Phys., vol. A99, no. 3, June 1967, pp. 442-456.
25. Poulet, M.; Michalowicz, A.; Le Guen, Y.; Kuroda, K.; Cronenberger, D.; and Coignet, G.: Le Faisceau de Protons Polarisés de 141 MeV du Synchrocyclotron D'Orsay. Application à la Mesure de la Polarisation dans la Diffusion Élastique p-<sup>12</sup>C et p-d. J. Phys., vol. 26, July 1965, pp. 399-402.
26. Kuroda, K.; Michalowicz, A.; and Poulet, M.: Mesure de la Polarisation dans la Diffusion Élastique Proton-Deuton à 155 MeV. J. Phys., vol. 26, June 1965, pp. 303-304.
27. Kuroda, K.; Michalowicz, A.; and Poulet, M.: Étude Expérimentale de la Diffusion Élastique Proton-Deuton à 155 MeV. Nucl. Phys., vol. 88, no. 1, Nov. 1966, pp. 33-44.
28. Adelberger, Rexford Earle: A Measurement of the Cross Section and Polarization in Proton-Deuteron Elastic Scattering at 198 MeV. Ph. D. Thesis, Univ. of Rochester, 1967.
29. Wilson, John W.: Comparison of Exact and Approximate Evaluations of the Single-Scattering Integral in Nucleon-Deuteron Elastic Scattering. NASA TN D-6884, 1972.
30. Sloan, Ian H.: Multiple-Scattering Analysis on a Soluble Neutron-Deuteron Model. Phys. Rev., second ser., vol. 185, no. 4, Sept. 20, 1969, pp. 1361-1370.
31. Abers, E. S.; Burkhardt, H.; Teplitz, V. L.; and Wilkin, C.: On the Shadow Effect in High-Energy Scattering From Deuterons. Nuovo Cimento, vol. XLII A, no. 2, Mar. 21, 1966, pp. 365-389.
32. Gross, Franz: Relativistic Treatment of Loosely Bound Systems in Scattering Theory. Phys. Rev., second ser., vol. 140, no. 2B, Oct. 25, 1965, pp. B410-B421.
33. Coleman, Ernest: Proton-Deuteron Elastic Scattering at High Momentum Transfers. Ph. D. Thesis, Univ. of Michigan, 1966.
34. Gourdin, M.; Le Bellac, M.; Renard, F. M.; and Van, J. Tran Thanh: Phenomenological Description of the Relativistic Neutron-Proton-Deuterium Vertex. Nuovo Cimento, vol. XXXVII, no. 2, May 16, 1965, pp. 524-542.

35. Blankenbecler, R.; and Cook, L. F., Jr.: Bound States and Dispersion Relations. Phys. Rev., second ser., vol. 119, no. 5, Sept. 1, 1960, pp. 1745-1752.
36. Goldberger, M. L.; Grisaru, M. T.; MacDowell, S. W.; and Wong, D. Y.: Theory of Low-Energy Nucleon-Nucleon Scattering. Phys. Rev., second ser., vol. 120, no. 6, Dec. 15, 1960, pp. 2250-2276.
37. Barut, A. O.: The Theory of the Scattering Matrix. Macmillan Co., c.1967.
38. Alt, E. O.; Grassberger, P.; and Sandhas, W.: Reduction of the Three-Particle Collision Problem to Multi-Channel Two-Particle Lippmann-Schwinger Equations. Nucl. Phys. B, vol. B2, no. 2, June 15, 1967, pp. 167-180.
39. Lassila, K. E.; Hull, M. H., Jr.; Ruppel, H. M.; McDonald, F. A.; and Breit, G.: Note on a Nucleon-Nucleon Potential. Phys. Rev., second ser., vol. 126, no. 2, Apr. 15, 1962, pp. 881-882.
40. Hamada, T.; and Johnston, I. D.: A Potential Model Representation of Two-Nucleon Data Below 315 MeV. Nucl. Phys., vol. 34, May 1962 - June 1962, pp. 382-403.
41. Reid, Roderick V., Jr.: Local Phenomenological Nucleon-Nucleon Potentials. Ann. Phys., vol. 50, No. 3, Dec. 1968, pp. 411-448.
42. Bressel, C. N.; Kerman, A. K.; and Rouben, B.: Soft-Core Nucleon-Nucleon Potential. Nucl. Phys., vol. A124, no. 3, Feb. 17, 1969, pp. 624-636.
43. Stapp, H. P.; Ypsilantis, T. J.; and Metropolis, N.: Phase-Shift Analysis of 310-Mev Proton-Proton Scattering Experiments. Phys. Rev., second ser., vol. 105, no. 1, Jan. 1, 1957, pp. 302-310.
44. MacGregor, M. H.; Moravcsik, M. J.; and Stapp, H. P.: Nucleon-Nucleon Scattering Experiments and Their Phenomenological Analysis. Annual Review of Nuclear Science, Emilio Segrè, Gerhart Friedlander, and Walter E. Meyerhof, eds., Vol. 10, 1960, pp. 291-352.

45. MacGregor, Malcolm H.; Arndt, Richard A.; and Wright, Robert M.: Determination of the Nucleon-Nucleon Scattering Matrix.
- VII. (p,p) Analysis From 0 to 400 MeV. Phys. Rev., second ser., vol. 169, no. 5, May 25, 1968, pp. 1128-1148.
- VIII. (p,p) Analysis From 350 to 750 MeV. Phys. Rev., second ser., vol. 169, no. 5, May 25, 1968, pp. 1149-1166.
- IX. (n,p) Analysis From 7 to 750 MeV. Phys. Rev., second ser., vol. 173, no. 5, Sept. 25, 1968, pp. 1272-1301.
46. Humberston, J. W.; and Wallace, J. B. G.: Deuteron Wave Functions for the Hamada-Johnston Potential. Nucl. Phys., vol. A141, no. 2, Feb. 2, 1970, pp. 362-368.
47. Wilson, J. W.: Comparison of Exact and Approximate Evaluations of the Single Scattering Integral in Nucleon-Deuteron Elastic Scattering. Bull. Amer. Phys. Soc., ser. II, vol. 17, no. 4, Apr. 1972, p. 439.
48. Glendenning, Norman K.; and Kramer, Gustav: Nucleon-Nucleon Triplet-Even Potentials. Phys. Rev., second ser., vol. 126, no. 6, June 15, 1962, pp. 2159-2168.
49. Jarvis, O. N.; and Rose, B.: Renormalization of p-p Polarization Data in the 50-150 MeV Energy Region. Phys., Lett., vol. 15, no. 3, Apr. 1, 1965, pp. 271-273.

TABLE I.- FITTED STRUCTURE PARAMETERS FOR GAUSSIAN APPROXIMATION  
OF HAMADA-JOHNSTON WAVE FUNCTION

$$\left[ \text{Normalization } \int_0^\infty [\varphi_0^2(q) + \varphi_2^2(q)] q^2 dq = 1; \varphi_L(q) = \sum_n a_{L_n} \exp(-b_{L_n} q^2) \right]$$

n	$a_{0_n}$ , amu <sup>-3/2</sup>	$b_{0_n}$ , amu <sup>-2</sup>	$a_{2_n}$ , amu <sup>-3/2</sup>	$b_{2_n}$ , amu <sup>-2</sup>
1	92.41770	204.15580	-1.683110	11.27690
2	12.74490	25.77230	-.971535	2.85898
3	-.35298	1.29835	2.659846	465.61200

TABLE II. - COMPARISON OF STICKING FACTORS OF APPROXIMATE GAUSSIAN  
WAVE FUNCTIONS WITH ANALYTIC HAMADA-JOHNSTON SOLUTION  
(REF. 46) AND THAT OF GLENDENNING AND KRAMER (REF. 48)

$ \vec{Q} $ , fm <sup>-1</sup>	S-state sticking factor, S(Q) <sub>0,0</sub>		
	H-W*	Approximate Gaussian	G-K <sup>†</sup>
0.0	0.930	0.928	0.926
.5	.797	.790	.795
1.0	.555	.526	.557
1.5	.356	.323	.361
2.0	.219	.201	.226
3.0	.071	.064	.078
4.0	.011	.007	.016
5.0	-.010	-.010	-.007
6.0	-.016	-.012	-.014

\*Humberston-Wallace L = 4, N = 2 analytic solution.

<sup>†</sup>Glendenning and Kramer values for potential number 9.

TABLE III.- FITTED PARAMETERS FROM ANALYSIS OF POSTMA AND WILSON EXPERIMENTS (REF. 18)

Method	$F(\hat{P})$	$N_{\sigma}$	P <sub>D</sub> , percent	S-wave		P-wave	
				D <sub>0</sub>	Q <sub>0</sub>	D <sub>1</sub>	Q <sub>1</sub>
Kottler-Kowalski (ref. 7)	124.0	1.012	6.56	0.004 - 0.015i	-0.009 + 0.010i	-----	-----
Asymptotic	134.8	1.012	5.69	0.008 - 0.008i	-0.013 + 0.007i	-----	-----
Numerical integration	208.2	1.012	5.16	0.018 + 0.008i	-0.015 + 0.011i	-----	-----
Numerical integration	48.8	0.902	3.93	0.0118 + 0.0008i	0.0050 + 0.0143i	-0.0147 - 0.0028i	0.0406 - 0.0101i
Numerical integration <sup>*</sup>	60.1	0.935	5.93	0.0113 + 0.0010i	0.0035 + 0.0129i	-0.0178 + 0.0032i	0.0404 - 0.0063i
Numerical integration <sup>†</sup>	55.1	0.924	5.43	0.0115 + 0.0010i	0.0037 + 0.0132i	-0.0170 + 0.0026i	0.0403 - 0.0073i

<sup>\*</sup> P<sub>D</sub> constrained to  $6.93 \pm 1.0$ .

<sup>†</sup> P<sub>D</sub> constrained to  $6.93 \pm 1.5$ .

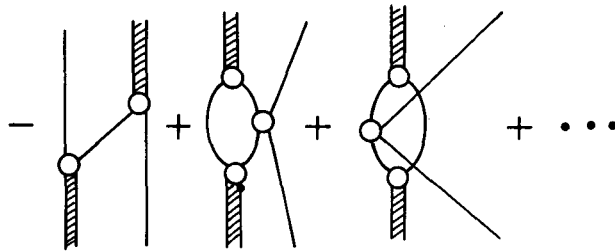


Figure 1.- Fully symmetrized multiple-scattering series in terms of renormalized fermion lines and proper vertex parts.

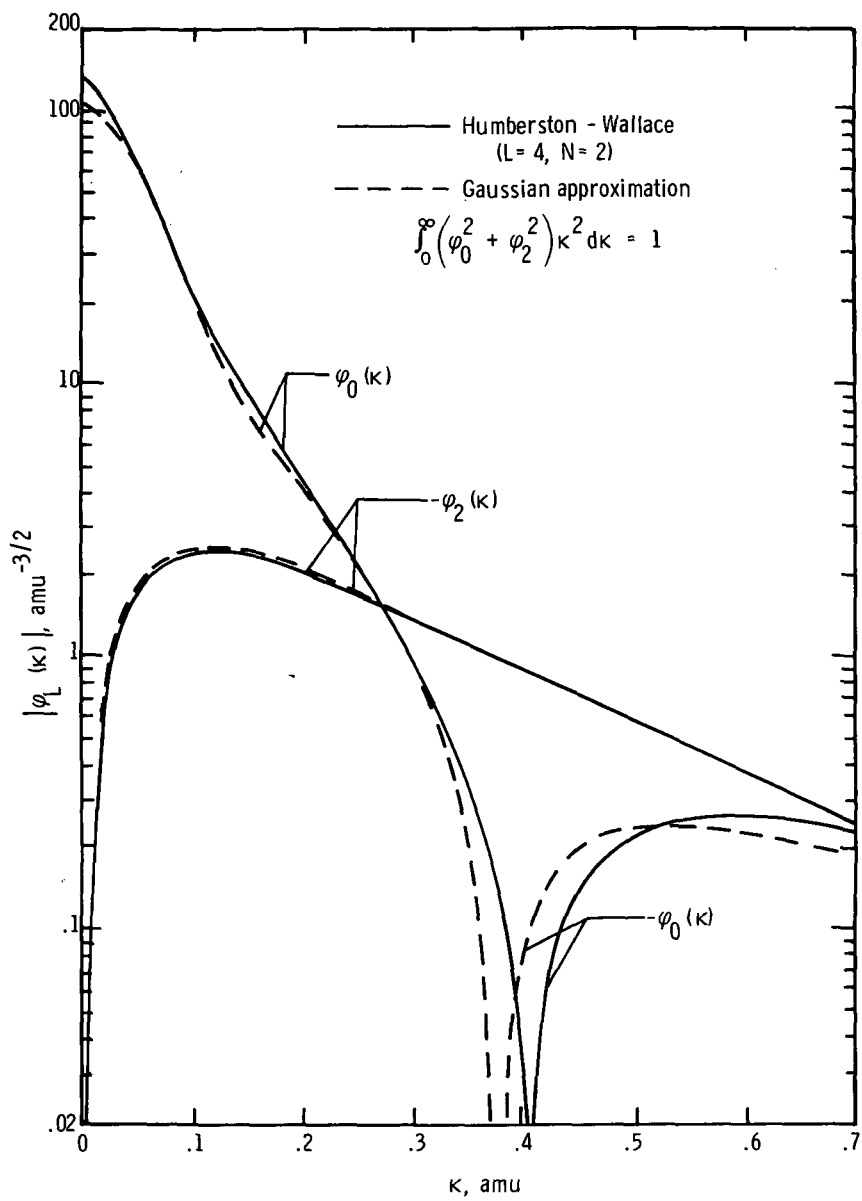


Figure 2.- Comparison of approximate Gaussian wave function with the Hamada-Johnston wave function obtained by Humberston and Wallace (ref. 46).

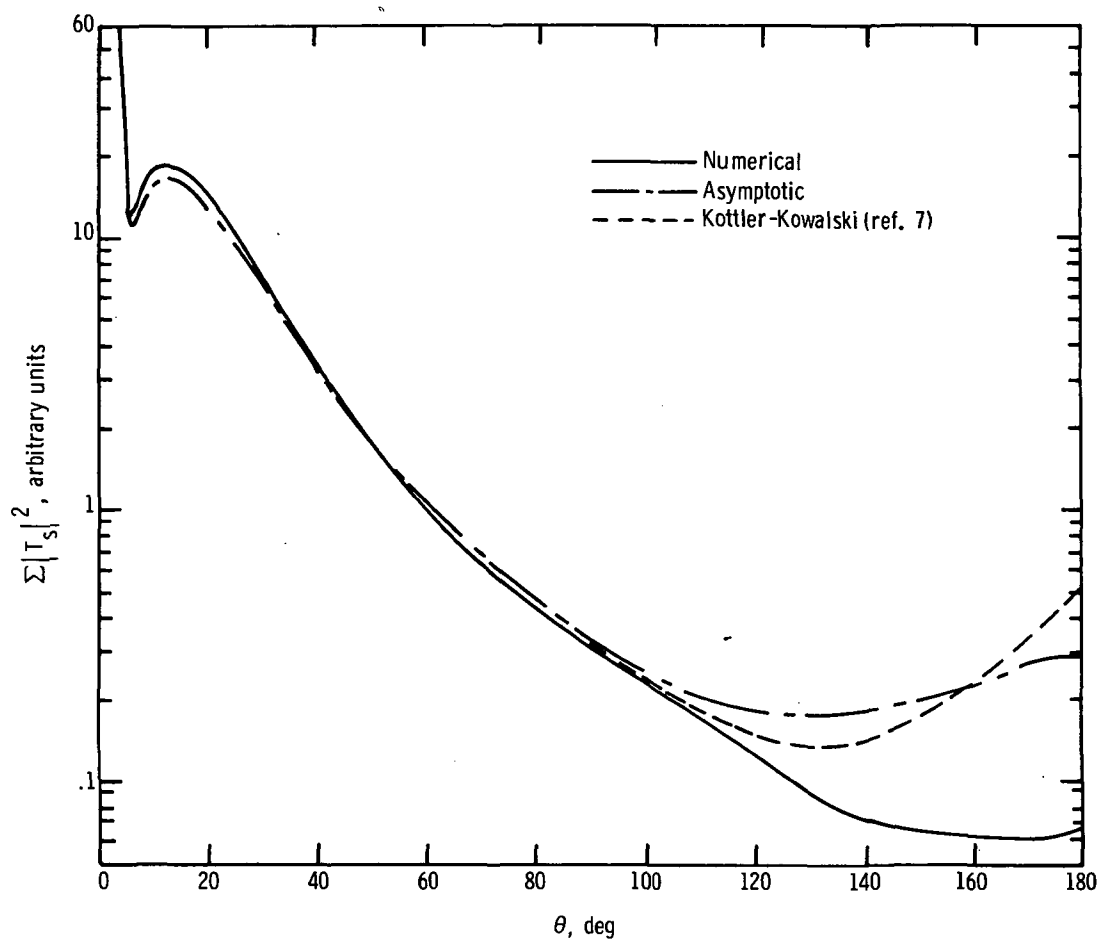
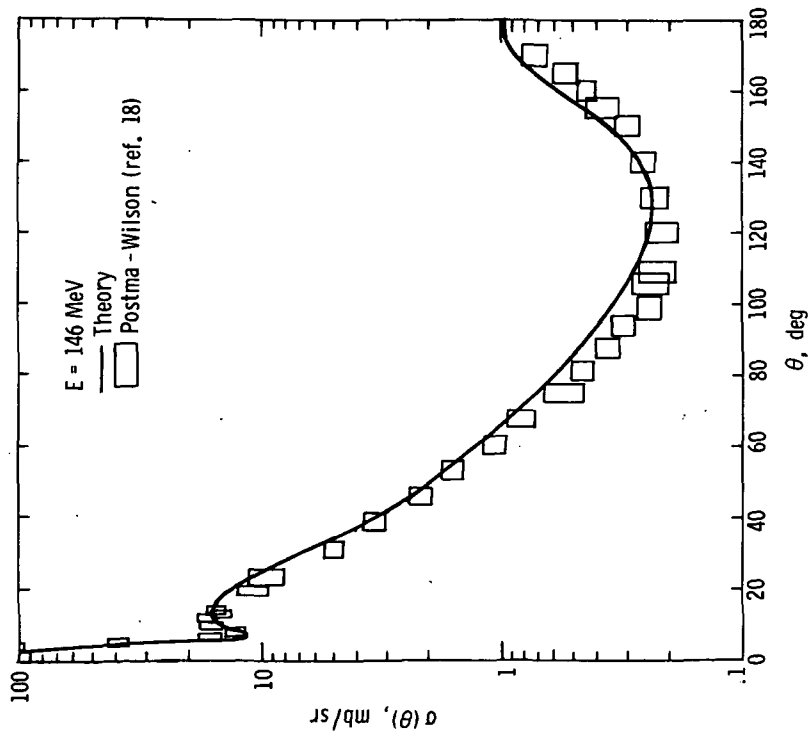
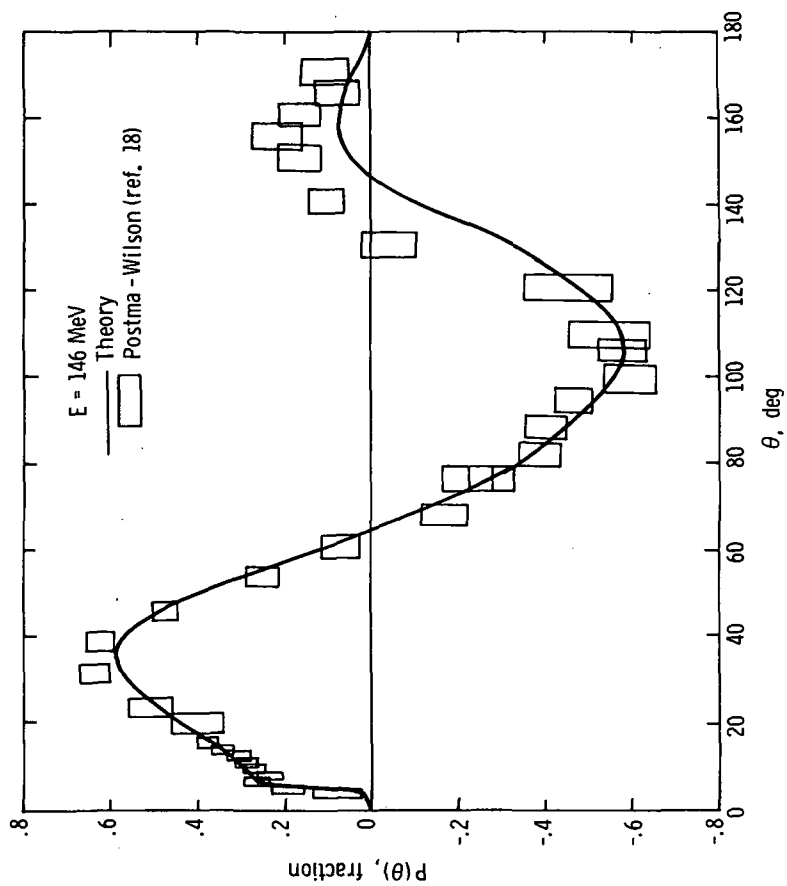


Figure 3.- Modulus squared of single-scattering integral as function of center-of-mass scattering angle for 146-MeV proton laboratory energy.



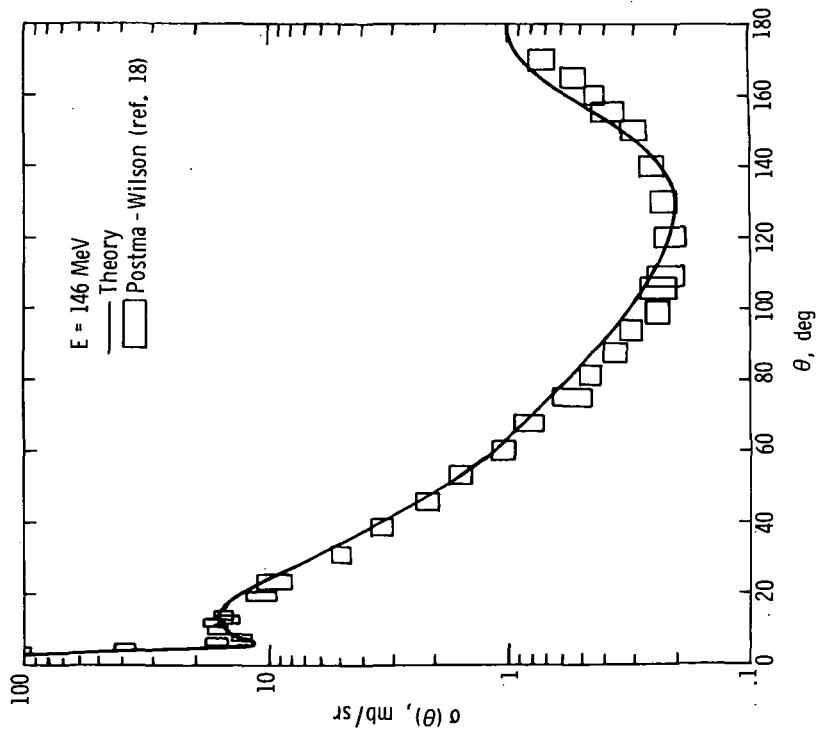


(a) Cross section.

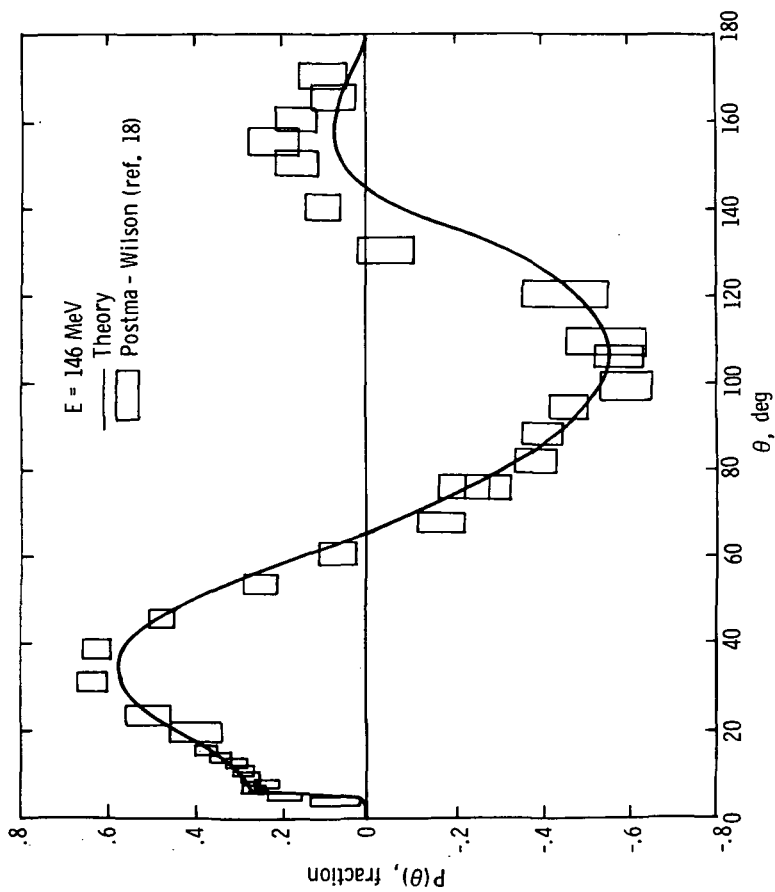


(b) Polarization.

Figure 4.- Comparison of S-wave fits using Kottler-Kowalski prescription (ref. 7) for single scattering. Best estimate of D-state percentage is 6.56.

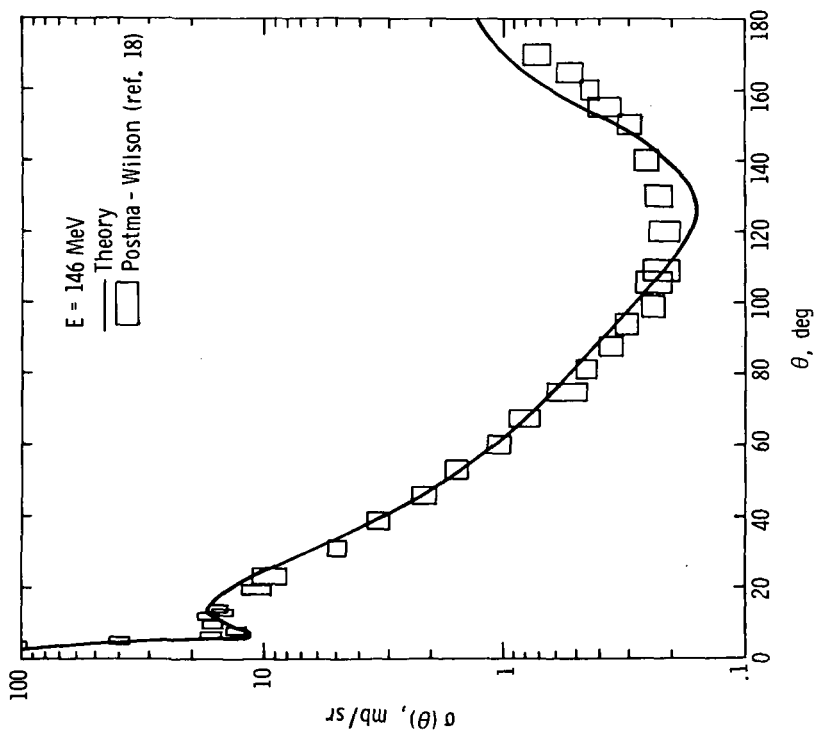


(a) Cross section.

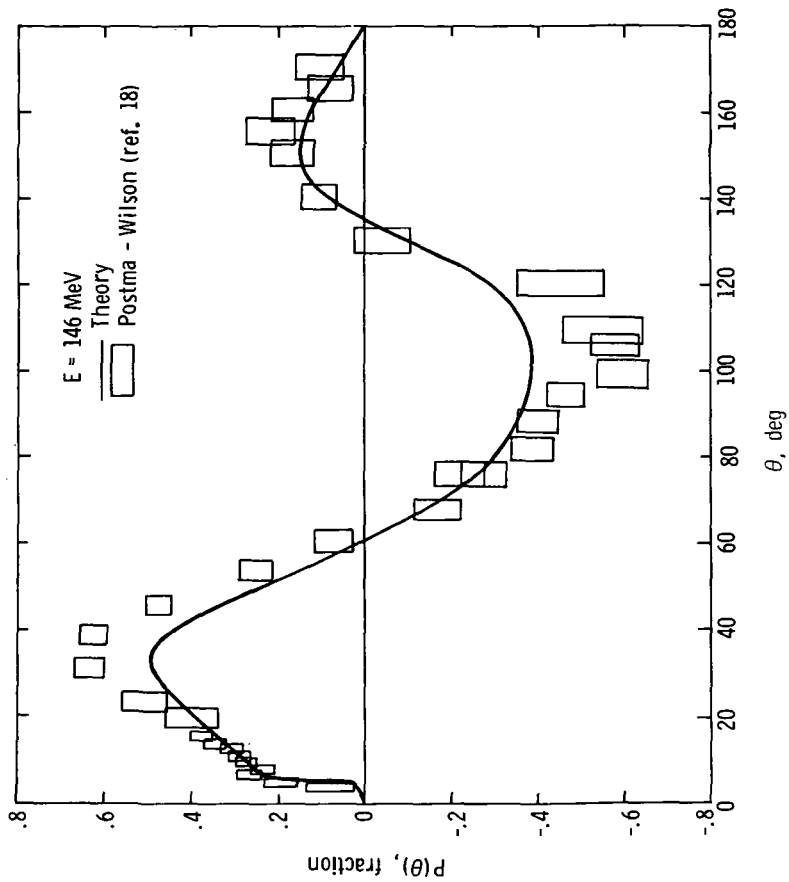


(b) Polarization.

Figure 5.- Comparison of S-wave fits using Asymptotic method to approximate single scattering. Best estimate of D-state percentage is 5.69.

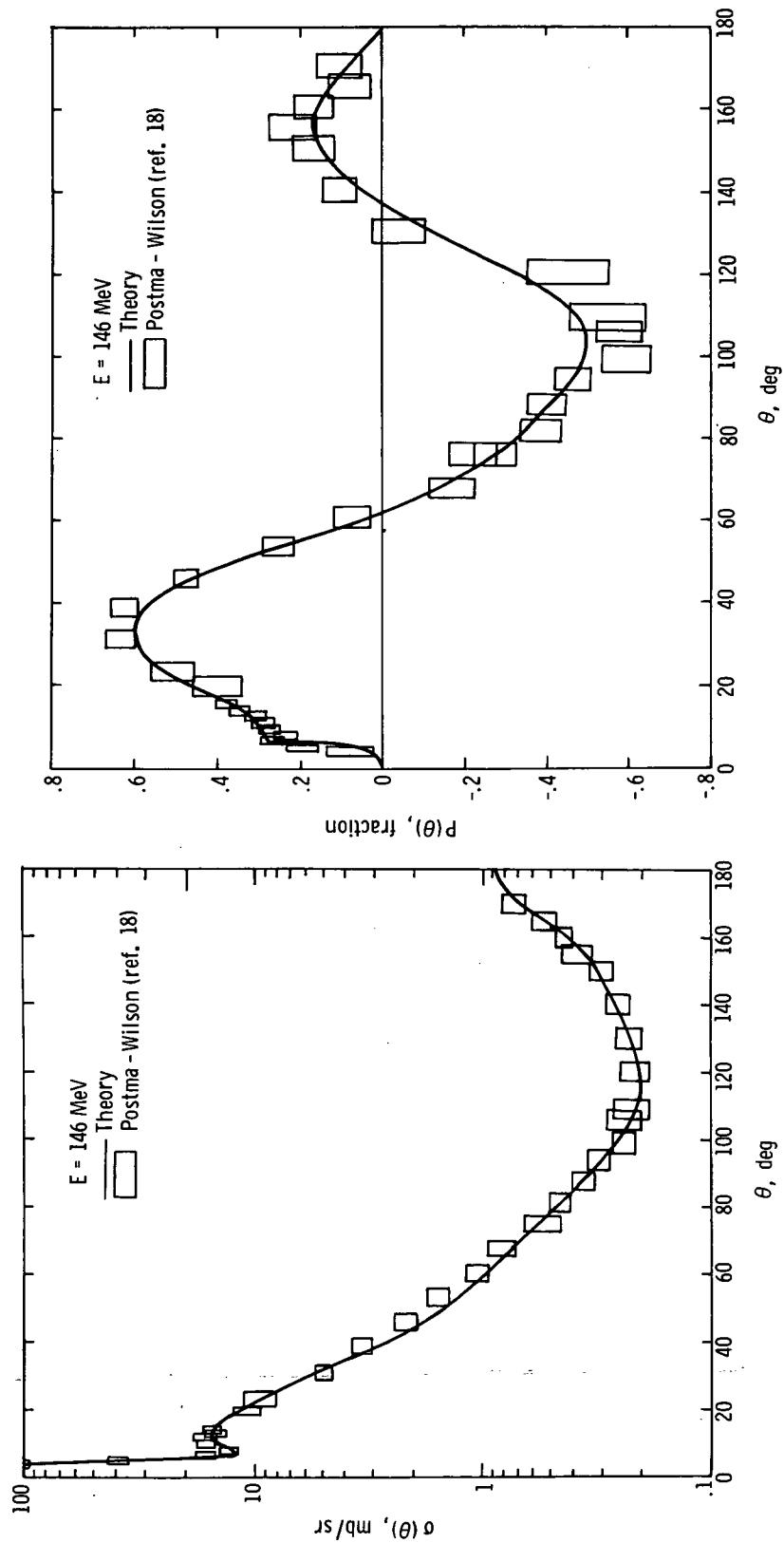


(a) Cross section.



(b) Polarization.

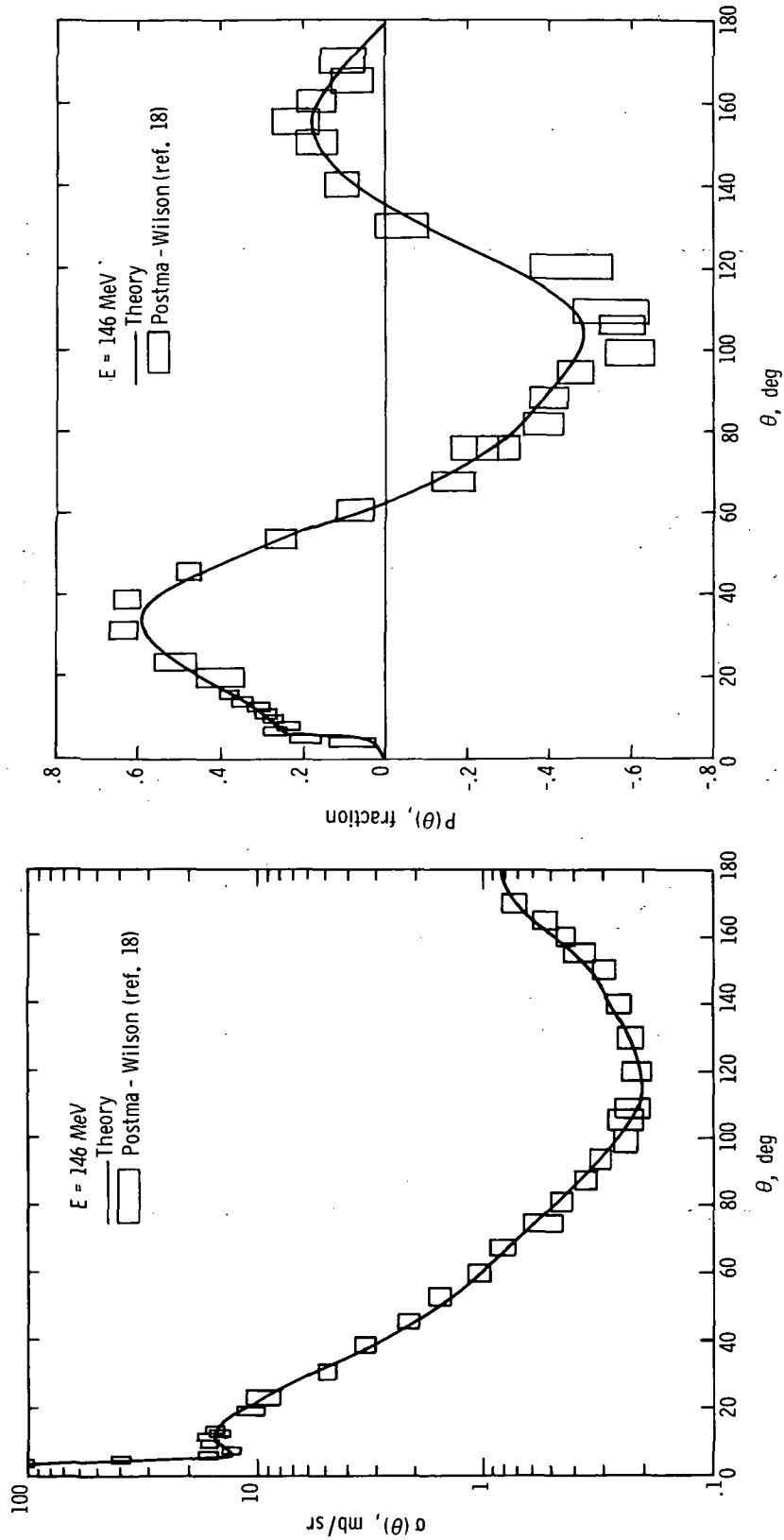
Figure 6.- Comparison of S-wave fits for numerical integration of single scattering.  
Best estimate of D-state percentage is 5.16.



(a) Cross section.

(b) Polarization.

Figure 7.- Comparison of S-wave plus P-wave fits using numerical integration of single scattering. Best estimate of D-state percentage is 3.93.



(a) Cross section.

(b) Polarization.

Figure 8.- Comparison of S-wave plus P-wave fits using numerical integration for single scattering.  
 D-state restricted to  $6.93 \pm 1$  percent in search procedures and 5.93 at minimum.

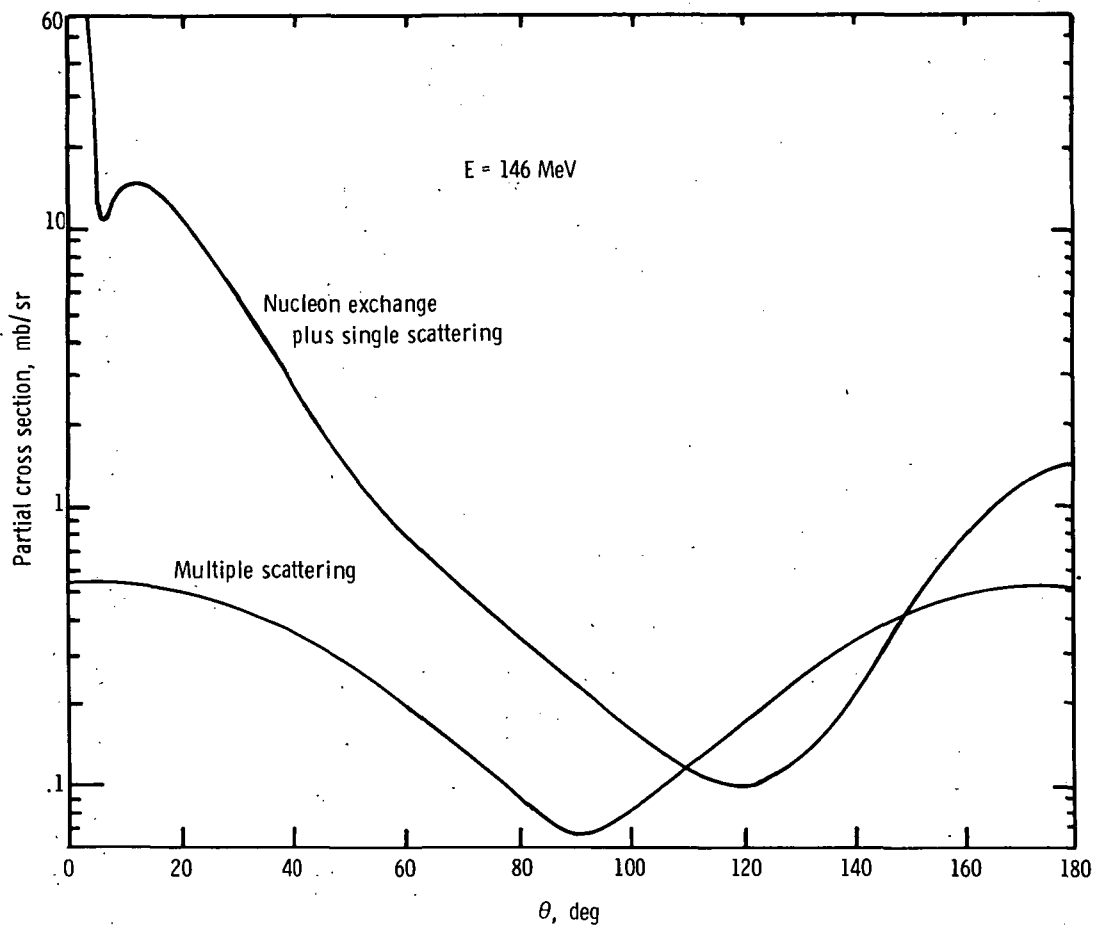


Figure 9.- Comparison of partial cross section of nucleon exchange plus single scattering to that produced by multiple scattering as estimated from experimental data.



POSTMASTER: If Undeliverable (Section 158  
Postal Manual) Do Not Return

*"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."*

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

## NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

**TECHNICAL REPORTS:** Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

**TECHNICAL NOTES:** Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

**TECHNICAL MEMORANDUMS:** Information receiving limited distribution because of preliminary data, security classification, or other reasons. Also includes conference proceedings with either limited or unlimited distribution.

**CONTRACTOR REPORTS:** Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

**TECHNICAL TRANSLATIONS:** Information published in a foreign language considered to merit NASA distribution in English.

**SPECIAL PUBLICATIONS:** Information derived from or of value to NASA activities. Publications include final reports of major projects, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

**TECHNOLOGY UTILIZATION PUBLICATIONS:** Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

*Details on the availability of these publications may be obtained from:*

**SCIENTIFIC AND TECHNICAL INFORMATION OFFICE**

**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION**

**Washington, D.C. 20546**